



UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

Escola d'Enginyeria de Telecomunicació
i Aeroespacial de Castelldefels

TREBALL DE FI DE GRAU

TFG TITLE: Design of a quaternion based ADCS for a small space telescope

DEGREE: Grau en Enginyeria d'Aeronavegació

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DATE: July 24, 2018

Títol: Disseny d'un ADCS basat en quaternions per a un petit telescopi espacial

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Data: 24 de juliol de 2018

Resum

Aquest projecte es centra en l'estudi previ al desenvolupament d'un Sistema de Determinació i Control d'Actitud (ADCS) per a un petit telescopi espacial, el Humble Space Telescope (hST). El sistema ADCS és essencial en l'operació del telescopi ja que ha de garantir una bona observació de les regions de l'espai d'interés. El sistema proposat es basa en l'ús de quaternions per tal d'alleugerar el pes computacional. L'algoritme QUEST (Quaternion Estimator) s'utilitzarà per a la determinació de l'actitud a partir de les observacions extretes pels sensors, principalment star trackers. El control es durà a terme mitjançant un controlador PD (Proportional Derivative) a través d'una llei de control basada en el quaternió error entre l'actitud desitjada i l'actual. L'estabilitat de la llei de control es verificarà a través d'un estudi de Lyapunov. Es durà a terme un estudi empíric sobre l'efecte dels guanyos proporcional i derivatiu del controlador en la resposta del sistema. El principal hardware per a la part de control seran reaction wheels afegint l'ús de magnetorquers per a la desaturació d'aquestes. Les simulacions numèriques que es realitzaran verifiquen la viabilitat del sistema proposat.

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Overview

This project focuses on the study previous to the development of an Attitude Determination and Control System (ADCS) of a small space telescope, the Humble Space Telescope (hST). The ADCS is essential for the telescope operation as it must guarantee a good observation of the space regions of interest. The proposed system is based on the use of quaternions to lighten the computational weight. The QUEST (Quaternion Estimator) algorithm will be used for the attitude determination from the vector observations made with the sensors, mainly star trackers. The control part will be done by means of a PD (Proportional Derivative) controller using a control law based on the error quaternion between the actual and the desired attitude. The stability of the proposed control law will be verified by a Lyapunov study. An empirical study of the effect of the proportional and derivative gains on the system's response will be carried out. The main hardware for the control part will be reaction wheels, combined with the use of magnetorquers for the desaturation of the reaction wheels. The numerical simulations that are going to be conducted will verify the viability of the proposed system.

CONTENTS

Introduction	1
CHAPTER 1. Background	3
1.1. What's attitude?	3
1.2. Attitude Geometry	3
1.3. Sensors	4
1.3.1. Earth Sensors	5
1.3.2. Sun Sensors	5
1.3.3. Star Trackers	6
1.3.4. Magnetometers	7
1.3.5. Gyroscopes	7
1.4. Actuators	7
1.4.1. Thrusters	8
1.4.2. Reaction and Momentum Wheels	8
1.4.3. Magnetorquers	9
1.5. Telescope Geometry	9
CHAPTER 2. Attitude representation	11
2.1. Attitude matrix	11
2.2. Euler angles	12
2.3. Quaternions	12
2.3.1. Properties and Basic Operations	13
2.3.2. Euler Angles to Quaternion	15
2.3.3. Quaternion to Attitude Matrix	15
CHAPTER 3. Rotational dynamics	17
3.1. Inertia tensor	17
3.2. Kinematics	17
3.3. Dynamics	18

CHAPTER 4. Attitude determination	21
4.1. Wahba's problem	21
4.2. QUEST algorithm	21
CHAPTER 5. Attitude control	25
5.1. Feedback Control	25
5.2. Disturbances	26
5.3. PID controller	27
5.3.1. Control Law	29
5.3.2. Lyapunov Stability test	29
CHAPTER 6. Results and Analysis	31
Conclusions	37
Future Work	39
Bibliography	41
APPENDIX A. Matlab codes	45
A.1. Main program	45
A.2. QUEST Algorithm function	48
A.3. Control function	50

LIST OF FIGURES

1.1 Celestial sphere reference frame.	4
1.2 Digital Sun Sensor (DSS) schematic [1]	6
1.3 Star tracker schematic [2]	6
1.4 Momentum wheel cross section (<i>Honeywell</i>)	8
1.5 First approach of the hST geometry.	10
2.1 Euler rotation sequence. [3]	12
5.1 Block diagram example.	25
5.2 Gravitational and aerodynamic torques as function of orbit altitude	28
5.3 PID controller Block Diagram	28
6.1 Simulations results for $k_p = 1$ and $k_d = 15$	32
6.2 Simulations results for $k_p = 10$ and $k_d = 15$	33
6.3 Simulations results for $k_p = 10$ and $k_d = 5$	34
6.4 Simulations results for $k_p = 1$ and $k_d = 25$	35

LIST OF TABLES

1.1	Potential accuracies by type of sensor	5
1.2	hST component dimensions and weights.	10
5.1	Environmental Disturbance Torques ([4] p.17)	26

INTRODUCTION

The contribution of large, space-based observatories to the advancement of astronomy cannot be overstated. Despite its importance, the number of space observatories is strictly limited by its high cost, and this small number limitation implies that they are terribly oversubscribed. Then, many observing projects do not receive any observing time, having to move towards ground-based observatories (the best of which are also heavily oversubscribed).

The goal of this project consists in designing the Attitude Determination and Control System (ADCS) for a small, very affordable space telescope that could be launched as a secondary payload in a large launcher, or as primary payload in an inexpensive one. Having several of these small space telescopes (hereafter referred to as hSTs, or Humble Space Telescopes) could allow many projects having their required observing time. These projects could be the ones requiring frequent observations –or even a dedicated instrument–, high-risk, high-reward projects, or pathfinders for advanced projects that would require larger instruments if the first observations proved promising; all this without forgetting other, more standard projects similar to the ones that use 1 – 3 meter telescopes on the Earth, and that are a fundamental part of the steady advance of Astronomy.

To be of use in this context, the hST must be very affordable: 2 million euros, excluding launch and insurances. The diameter of the hST should be in the range of 0.5 to 0.75 m, with a Ritchey-Chretien optical system that will give it a high compactness. To further reduce its form factor during the launch, the structure will be extendable, and to further reduce its cost, 3D printed with space-qualified materials.

Developing countries are one of the main focus of this project. For the cost of a small observatory on the ground, they would get first-hand experience in space technology, as well as a space facility that could boost local Astronomy. After gaining experience with this kind of satellites, they could move into more ambitious, regional or international programs.

Focusing on the ADCS, the control system must focus on the use of momentum wheels as primary source for control and magnetorquers to unload the wheels when saturated. Attitude acquisition would be based on a star tracker, the feedback given by the solar panels (Sun's relative position) and a Sun sensor pointing coaxially with the main telescope. The data collected from the different sensors will be processed using the QUEST algorithm and the control torque computed with a PD controller with quaternion-based feedback.

CHAPTER 1. BACKGROUND

1.1. What's attitude?

Attitude is defined as the orientation of a rigid body, in this case a spacecraft, relative to some reference coordinate frame and the spacecraft's motion about its center of mass. The kind of orientation required depends on the payload; in our case, a Space Telescope, it is obvious that the payload must be pointed to specific regions of the sky to perform scientific research on its target, and that it must be aiming at this target for a period of time long enough to allow good signal-to-noise ratio data.

This kind of attitude control is called 3-axis attitude, as the reference system tied to the satellite must be correctly oriented with regard to an external reference system (which, for the purposes of this mission, will be tied to the celestial sphere). Naturally, the ADCS system must be composed of a set of sensors to determine the attitude, and a set of actuators to control it. It is not the purpose of this work to deal with the actual hardware, but we will include in sections 1.3. and 1.4. a succinct description of the main sensors and actuators that could be employed for the hST.

There are other considerations that must be accounted for. Attitude determination can be assisted by the Electric Power Subsystem: as the solar panels will be fixed to the body of the satellite (see section 1.5. and Fig.1.5), the direction towards the Sun can be roughly determined by measuring the power generated. While this does not give a very accurate attitude determination, it can be used in case of sensor malfunction, and to ensure that the telescope will not be entering the solar avoidance zone (see requirement 3).

In a nutshell, controlling the attitude of the telescope is the essential requirement to be able to perform a good operation.

1.2. Attitude Geometry

For this work, there are two reference systems of paramount importance, the body system (referred to as B), and the inertial reference system tied to the celestial sphere (referred to as I). While it is called "inertial", the celestial reference system has small accelerations due to the motions of the Earth (precession, nutation...). Then, the reference system is updated every few years (and then is tied to different epochs) and for the sake of detailed calculations is updated to the epoch of observation. We will not deal with such technicalities that belong to the astronomical realm.

The body reference system will be made with an axis in the direction of the optical axis of the telescope (z axis), an axis pointing towards the Sun (x axis), and a third axis to conform a direct reference system (y axis). The inertial reference system will have an axis pointing

towards the vernal equinox (x axis), an axis pointing to the celestial North pole (z axis), and a third axis as required to conform a direct reference system (y axis).

These simple definitions are enough for the work presented here. Once the design of the telescope is refined, the body reference system could experience small refinements, but the general layout is unlikely to change.

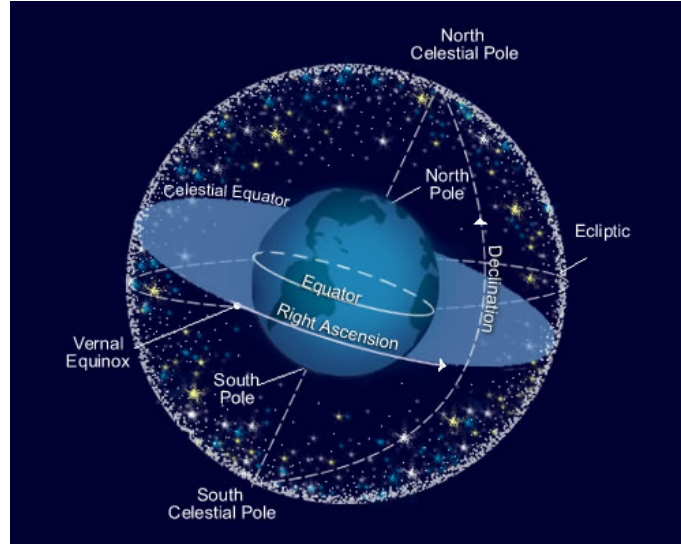


Figure 1.1: Celestial sphere reference frame.

1.3. Sensors

Sensors are those components used to determine the attitude of the spacecraft by means of observations of some external reference points or by measuring the changes in the state of rotation of the satellite. There exist several types of sensors depending on how each one computes the attitude of the satellite and which reference it takes, being the sun, the stars or the earth location among others. A brief introduction to each sensor category is given below in this section.

We will not deal with the errors associated with measures, as they are dependent on the specific device used. Nevertheless, these errors must be self-consistently taken into account, as the uncertainty in the measurements directly translates into an uncertainty in the attitude determination which, on its turn, will translate into an uncertainty on the required control torque. Table 1.1 gives a rough indication of the expected accuracies for the different kinds of current attitude sensors.

Sensor type	Potential accuracy
Star Trackers	1 arc-second
Gyroscopes	10 arc-seconds
Sun Sensors	1 arc-minute
Earth Sensors (Horizon)	6 arc-minutes
Magnetometers	30 arc-minutes

Table 1.1: Potential accuracies by type of sensor

1.3.1. Earth Sensors

Earth Sensors or Horizon Sensors detect different points of the Earth's horizon at infrared wavelengths and are widely used on Earth-orbiting spacecraft, especially on those pointing to Earth. More advanced sensors can use thermopiles to detect the difference of temperature between the equator and the poles.

There are two types of Earth Sensors, *static* and *scanning* sensors. Scanning sensors can use either a rotational or oscillatory scan with a small FOV detector across the Earth. Static ones are limited to Earth-pointing spacecraft in a certain range of altitude.

This kind of sensor is not well-suited for the attitude determination of the satellite, but could be used to ensure that the telescope does not point into the avoidance zone around the Earth (requirement 4).

1.3.2. Sun Sensors

Sun Sensors measure the orientation of the spacecraft relative to the Sun. Those can be classified into two categories, Coarse or Analog Sun Sensors (CSS) and Fine or Digital Sun Sensors (DSS). The attitude is obtained as function of a current output proportional to the intensity of the incident light. Because of the presence of incoming light from the Earth's albedo and glint Sun Sensors must be calibrated to be able to treat the Sun as a point source. While CSS provide non-directional cosine information and a minimum of 6 are required to get full attitude information, DSS provide 2-axis estimation of Sun's location and thus only 4 are required (Fig1.2).

A Sun sensor, as already commented, could be used for rough determination of the solar avoidance zone, as well as for pointing the solar panels in case of partial malfunction of the attitude determination subsystem. In any case, as also commented before, solar panels could be used as a very coarse Sun sensor (probably with an accuracy of $5\text{-}10^\circ$).

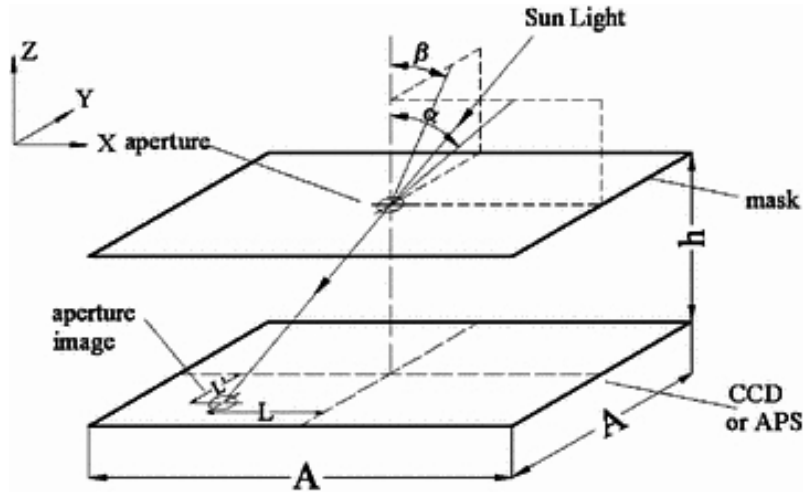


Figure 1.2: Digital Sun Sensor (DSS) schematic [1]

1.3.3. Star Trackers

Star Trackers measure multiple star coordinates simultaneously in the spacecraft frame using a digital camera with a CCD (Charge-Coupled Device) or CMOS (Complementary Metal-Oxide Semiconductor) sensor and determine the attitude by comparing the observed coordinates with an internal star directions catalog (Fig.1.3). Star sensors are the most accurate, reaching accuracies of arc-seconds. CCD sensors present lower noise but CMOS are more resistant to radiation and are capable of reading different pixels at different rates.

Stars trackers will be the main attitude determination sensor for the mission, as they are the only ones that provide enough accuracy for fine pointing the telescope. A synergy can be obtained with the scientific payload, that can act as an extremely high accuracy star tracker for final pointing operations. Given the small field of view of the telescope, a subset of a detailed catalog (as the Gaia DR2 catalog, which includes a billion objects with very precise astrometry) should be uploaded with the pointing information.

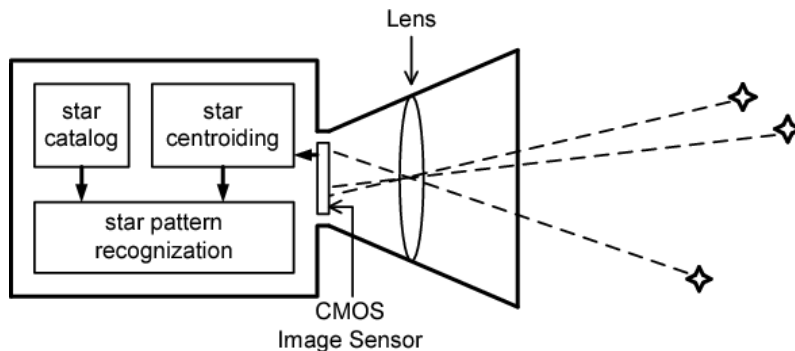


Figure 1.3: Star tracker schematic [2]

1.3.4. Magnetometers

Magnetometers are sensors that provide direction and magnitude of the Earth's magnetic field. Those can be *Quantum magnetometers* using basic atomic properties or *Induction magnetometers* following Faraday's law for magnetic inductance.

Magnetometers are interesting because of their low weight and power requirements and because they can operate in a wide range of temperatures but as the magnetic field is not totally known the accuracy is not high. Furthermore, the use of magnetometers is limited to low orbits as the Earth's magnetic field is reduced quadratically with distance.

While the final orbit has not yet been decided, it will very likely be over 1000 km in altitude, and then the magnetic field will be too weak and its value too uncertain to provide useful attitude information.

1.3.5. Gyroscopes

Gyroscopes, or simply Gyros, are instruments used as sensors to determine the spacecraft's orientation by measuring angular rates (*rate gyros*) or directly angular displacements (*rate-integrating gyros*).

In principle they can provide a very good accuracy for attitude determination, but as the measurements must be integrated with respect to time, the errors build up and must be zeroed periodically. To do so, it is necessary to obtain attitude information from external references, as the one gathered by Sun sensors or star trackers. Hence, we will not consider the use of gyroscopes to provide attitude references.

However, gyroscopes could also be used as actuators to generate torques and control the attitude of the spacecraft. Those are called *control moment gyros (CMGs)*.

1.4. Actuators

Once the attitude of the spacecraft is well-known by the determination phase, a control torque has to be applied to bring the actual attitude to the desired one. This is done using some of the actuators explained below.

Again, actuators are not error-free, as are not sensors, and then the expected accuracy should be checked along the maneuvering operations. As we have not selected any specific device, we will assume perfect (or error-free) actuators.

1.4.1. Thrusters

Thrusters or gas jets generate thrust by ejecting propellant in the opposite direction and torque by using pairs of thrusters. This thrust can be generated by accelerating ionized particles (*ion jets*) or by chemical reaction (*gas jets*).

Gas jets can be classified into *hot gas* if the energy is obtained from a chemical reaction or *cold gas* if it is from the latent heat of a phase change or the work compression when there is no phase change.

Futhermore, the propellant can be also classified into *monopropellant* or *bipropellant* depending on the number of components used. However, given that thrusters will not be used in our spacecraft we will not enter into a further explanation.

1.4.2. Reaction and Momentum Wheels

Reaction wheels are the most common actuators used as primary control in three-axis stabilised satellites. Full three-axis attitude control requires a minimum of three wheels (one per axis) although usually an extra wheel is used to get redundancy in case of failure, as it is more common than desired, or to generate a greater torque and momentum storage capability. If there are four momentum wheels, they are distributed with their rotation planes forming a tetrahedron, in such a way than a faulty gyroscope can always be substiytuted by the spare one.

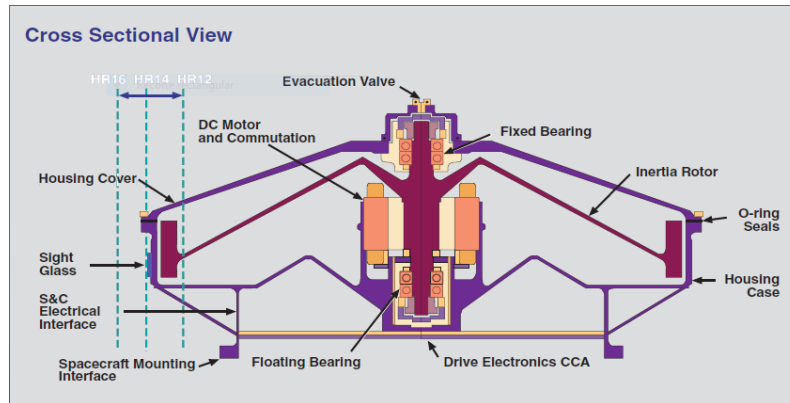


Figure 1.4: Momentum wheel cross section (*Honeywell*)

Reaction wheels consist on a wheel with an internal electric motor that controls the wheel's spinning velocity which determines the torque applied at each moment. Although they are good actuators for achieving good pointing accuracy, they are as well a source of disturbances caused by themselves. However, as the disturbance torque created by the wheels follows $\mathbf{r} \times \mathbf{F}$, its effect can be minimized by placing the wheels as closest to the spacecraft's center of mass as possible.

Eventually momentum wheels acquire their maximum rotational speed and must be braked without altering the rotational state of the rest of the satellite. This procedure is called momentum dumping (as the total angular momentum of the satellite must be reduced), and requires the use of an actuator that can release the angular moment, like thrusters or magnetorquers (see next section).

1.4.3. Magnetorquers

Magnetorquers are actuators based on the generation of magnetic dipoles to produce torque for angular momentum control.

The action law is

$$\vec{\tau} = \vec{\mu} \times \vec{B} \quad (1.1)$$

where $\vec{\mu}$ is the magnetic dipole of the magnetorquer and \vec{B} is the Earth's magnetic field. As commented for the magnetometer, the altitude will be excessive for the efficient use of magnetorquers. Even though, magnetorquers will be employed for momentum-wheel angular momentum dumping.

1.5. Telescope Geometry

A preliminary approach on the geometry of the hST is given in Fig.1.5. It consists of three main blocks, the lower box containing all the instruments, the main telescope itself and two solar panels to get the necessary power to operate the spacecraft.

The box is $90 \times 90 \times 50 \text{ cm}^3$ and, for the purposes of this project, is considered homogeneous for sake of simplicity. It would contain all scientific instrumentation as well as the bus of the spacecraft.

The telescope itself would be a Ritchey-Chrétien optical system with a diameter of 75cm plus a wall of 15mm of thickness and with a cylindrical shape of 175cm length.

For the power system, the two solar panels would be of $185 \times 65 \text{ cm}^2$ and would be installed with an angle of 165 degrees between them as in Fig.1.5. Besides being the source of power for the spacecraft, the solar panels located at one side of the telescope would be used as a shade to reduce incoming Sun's light to the telescope focal.

Every space mission starts by writing the set of requirements to be fulfilled by the spacecraft. In our case we will only list those related to the ADCS subsystem:

1. The satellite will be able to point to any point in the celestial sphere with an accuracy of less the 5 arcseconds and to keep its pointing with a drift smaller than 1 arcsecond per hour.

System	Dimensions (cm)	Weight (kg)
Homogeneous box	$90 \times 90 \times 50$	28
Telescope	$175 \times \varnothing 75$	120
Solar Panels (x2)	$185 \times 65 \times 1.5$ (each)	31 (each)
TOTAL		210

Table 1.2: hST component dimensions and weights.

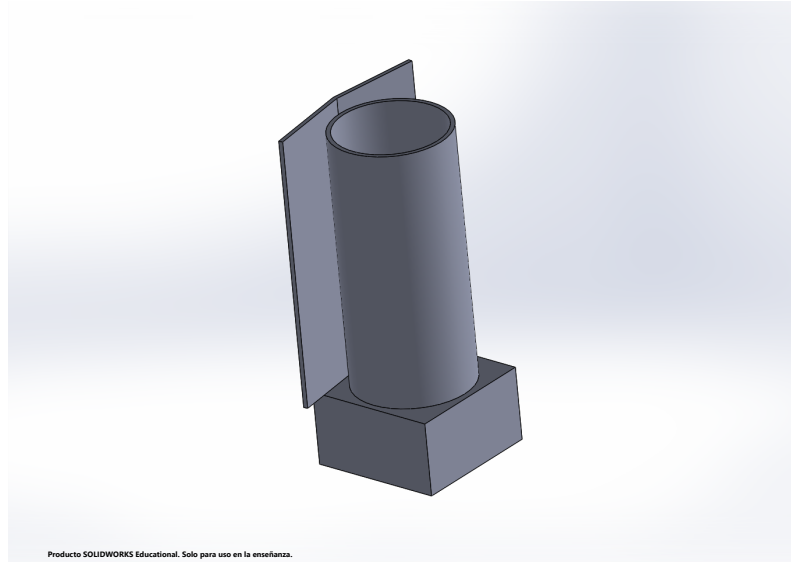


Figure 1.5: First approach of the hST geometry.

2. The solar panels will always be faced towards the Sun within an accuracy of 10 degrees
 3. The telescope will never point to a position less than 90 degrees apart of the Sun
 4. The telescope will never point to a position less than 60 degrees apart of the Earth
 5. The telescope will never point to a position less than 30 degrees apart of the Moon
- All these requirements must be accomplished simulatenously.

CHAPTER 2. ATTITUDE REPRESENTATION

We will consider the spacecraft as a rigid body in space with three orthogonal axis $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$ fixed in the spacecraft's body such that

$$\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{w}} \quad (2.1)$$

Then the problem of determining the attitude consists on determining this axis orientation relative to a reference coordinate system, which requires to specify 9 parameters organized in a 3×3 matrix of attitude A

$$A = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \quad (2.2)$$

2.1. Attitude matrix

Matrix A , referred to as *Attitude Matrix* or *Direction Cosines Matrix (DCM)* (2.2), consists on 9 elements that completely define the attitude of a rigid body in space. Each of its 9 components defines the cosine of the angle between the body unit vector and each of the reference axis; *i.e.*, each of the entries of the matrix is the scalar product between the vectors defining the reference system attached to the satellite and the unit vectors defining the inertial reference system.

Moreover, A is constrained by the fact that the body vectors $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$ are orthonormal and hence, by definition

$$u_1^2 + u_2^2 + u_3^2 = 1$$

$$v_1^2 + v_2^2 + v_3^2 = 1$$

$$w_1^2 + w_2^2 + w_3^2 = 1$$

and

$$u_1 v_1 + u_2 v_2 + u_3 v_3 = 0$$

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0$$

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0$$

Leading to the property that matrix A multiplied by its transpose results in the identity matrix

$$AA^T = I_{3 \times 3}$$

Finally, because of the right-handed triad (2.1) the determinant is $\det(A) = 1$ and matrix A is a *proper real orthogonal matrix*.

In terms of interpretation, matrix A is a transformation matrix that transforms a vector in the reference frame to the body frame coordinates. As A is a proper real orthogonal matrix it conserves the length of the transformed vector and thus $A\mathbf{a}$ represents a rotation of the vector \mathbf{a} to the body frame $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$.

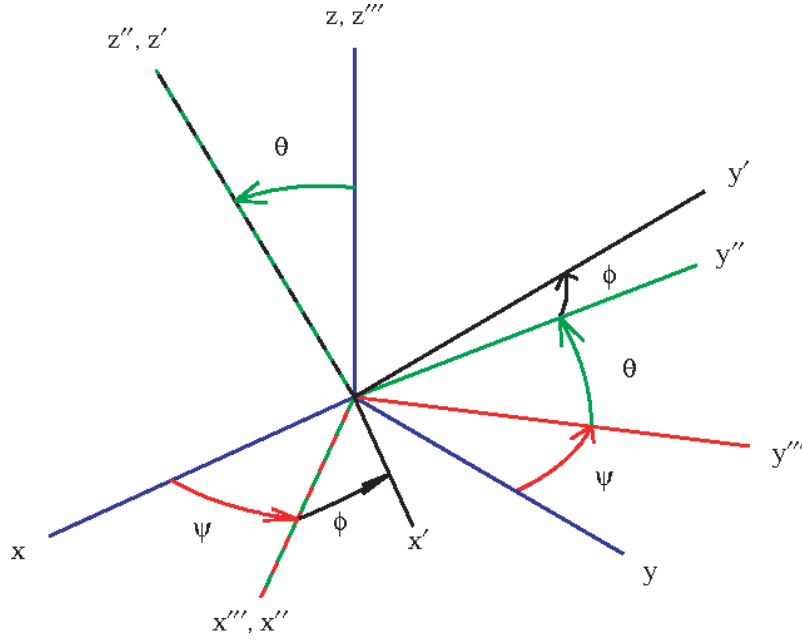


Figure 2.1: Euler rotation sequence. [3]

2.2. Euler angles

Any rotation in \mathbb{R}^3 can be described as a sequence of three coordinate rotations as follows, first ψ about k-axis (referred to as 3), then θ about i-axis (1) and finally ϕ about k-axis again (3) (Fig.2.1). Thus the rotation matrix is the matrix product of the three separate rotations (Eq.2.4).

$$R_{313}(\phi, \theta, \psi) = R_3(\phi)R_1(\theta)R_3(\psi) \quad (2.3)$$

$$R_{313}(\phi, \theta, \psi) = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & \cos\psi\sin\phi + \cos\theta\sin\psi\cos\phi & \sin\theta\sin\psi \\ -\sin\psi\cos\phi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \quad (2.4)$$

Euler angles have been widely used for attitude description, but have the problem that there can be found different rotation sequences that transform the system from the original attitude to the final one, a problem which is called *degeneracy*.

From a practical point of view, Euler angles require an important amount of calculations to deal with the attitude matrix, and then they are computationally expensive.

2.3. Quaternions

In this section we will see a brief introduction to quaternions and its use to define rotations in 3-dimensions such as defining the attitude of a satellite.

Invented by Hamilton in 1843, quaternions are hyper-complex numbers of rank 4 following the rule $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. The quaternion q is formed by four numbers divided into a scalar part and a vector part

$$q = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 + q_4$$

being $\mathbf{q} = [q_1 \ q_2 \ q_3]$ the vector part and q_4 the scalar part of the quaternion resulting in

$$q = \mathbf{q} + q_4$$

Quaternions transform a vector into a different one, thus changing its orientation as well as its length. In their application to attitude problems, the change in the norm of the vector is unimportant, and then only quaternions of unit norm are used:

$$||q||^2 = q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

In the case of quaternions, the composition of attitude changes is dealt with as algebraic equations, and then they are much less computationally intensive than Euler angles. Also, quaternions do not have any degeneracy in rotations of up to 720 degrees, and thus can give unique representations for attitude changes (that involve rotations of, at most, 360 degrees).

2.3.1. Properties and Basic Operations

Here we will describe the basic and most important operations and properties of quaternions.

- **Multiplication**

The product of a quaternion q and a scalar a is simply the product of a in each of the quaternion components $aq = \mathbf{i}aq_1 + \mathbf{j}aq_2 + \mathbf{k}aq_3 + aq_4$.

The product of two quaternions is a little bit more complex and must satisfy the following rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$$

$$\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$$

$$\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$$

Given the quaternions q and p , their product is obtained by

$$pq = p_4q_4 - \mathbf{p} \cdot \mathbf{q} + p_4\mathbf{q} + p_4\mathbf{q} + \mathbf{p} \times \mathbf{q} \quad (2.5)$$

which gives another quaternion as a result with scalar part

$$p_4q_4 - \mathbf{p} \cdot \mathbf{q}$$

and vector part

$$p_4\mathbf{q} + p_4\mathbf{q} + \mathbf{p} \times \mathbf{q}$$

Since the cross product $\mathbf{p} \times \mathbf{q}$ is not commutative neither is the quaternion product.

- **Conjugate of the quaternion**

The conjugate of a quaternion q , expressed by q^* is given by a change of the sign of the vector part, thus, given $q = \mathbf{q} + q_4$ its conjugate is $q^* = -\mathbf{q} + q_4$.

The sum of a quaternion and its complex gives the scalar

$$q + q^* = (q_4 + \mathbf{q}) + (q_4 - \mathbf{q}) = 2q_4 \quad (2.6)$$

- **The Norm**

The norm of a quaternion expresses its length and is defined by

$$N(q) = ||q|| = \sqrt{(qq^*)} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2} \quad (2.7)$$

From now on, to work with quaternions to express attitude we will work with normalized or unit quaternions, hence $||q|| = 1$.

- **Inverse of the quaternion**

The inverse of a quaternion q^{-1} is a quaternion such that

$$q^{-1}q = qq^{-1} = 1$$

and is given by

$$q^{-1} = \frac{q^*}{|q|^2} \quad (2.8)$$

which for unit quaternions is simply

$$q^{-1} = q^*$$

- **Geometric interpretation**

Relating unit quaternions with $|q_4|^2 + |\mathbf{q}|^2 = 1$ with Pitagoras Theorem $\cos^2 \theta + \sin^2 \theta = 1$ we can express quaternions in the form of

$$q = \hat{\mathbf{e}} \sin(\theta) + \cos(\theta) \quad (2.9)$$

which represents a rotation of 2θ about the axis represented by $\hat{\mathbf{e}}$.

And the rotation is represented as

$$\mathbf{v}' = q\mathbf{v}q^*$$

- **Derivative of the Quaternion**

The derivative of a quaternion is itself a quaternion.

$$\frac{dq}{dt} = \lim_{\Delta t \rightarrow 0} \frac{q(t + \Delta t) - q(t)}{\Delta t} \quad (2.10)$$

2.3.2. Euler Angles to Quaternion

Being ψ, θ, ϕ the Euler angles for rotations about z, y, x axis respectively, the associate quaternion $q = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 + q_4$ is obtained by:

$$q_1 = \cos \frac{\psi}{2} \cos \frac{\theta}{2} \sin \frac{\phi}{2} - \sin \frac{\psi}{2} \sin \frac{\theta}{2} \cos \frac{\phi}{2} \quad (2.11)$$

$$q_2 = \cos \frac{\psi}{2} \sin \frac{\theta}{2} \cos \frac{\phi}{2} + \sin \frac{\psi}{2} \cos \frac{\theta}{2} \sin \frac{\phi}{2} \quad (2.12)$$

$$q_3 = \sin \frac{\psi}{2} \cos \frac{\theta}{2} \cos \frac{\phi}{2} - \cos \frac{\psi}{2} \sin \frac{\theta}{2} \sin \frac{\phi}{2} \quad (2.13)$$

$$q_4 = \cos \frac{\psi}{2} \cos \frac{\theta}{2} \cos \frac{\phi}{2} + \sin \frac{\psi}{2} \sin \frac{\theta}{2} \sin \frac{\phi}{2} \quad (2.14)$$

2.3.3. Quaternion to Attitude Matrix

The Attitude Matrix A in terms of quaternions is expressed as follows

$$A = \begin{bmatrix} 2q_4^2 - 1 + 2q_1^2 & 2q_1q_2 + 2q_4q_3 & 2q_1q_3 - 2q_4q_2 \\ 2q_1q_2 - 2q_4q_3 & 2q_4^2 - 1 + 2q_2^2 & 2q_2q_3 + 2q_4q_1 \\ 2q_1q_3 + 2q_4q_2 & 2q_2q_3 - 2q_4q_1 & 2q_4^2 - 1 + 2q_3^2 \end{bmatrix} \quad (2.15)$$

or equivalently

$$A = \begin{bmatrix} q_4^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 + 2q_4q_3 & 2q_1q_3 - 2q_4q_2 \\ 2q_1q_2 - 2q_4q_3 & q_4^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 + 2q_4q_1 \\ 2q_1q_3 + 2q_4q_2 & 2q_2q_3 - 2q_4q_1 & q_4^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \quad (2.16)$$

CHAPTER 3. ROTATIONAL DYNAMICS

Within this chapter we are going to introduce the governing equations for the rotation of the spacecraft about its center of mass and the different parameters that affect it. First of all we are going to propose a first approach of the satellite's configuration and thus its inertia tensor, and then we are going to move towards the presentation of the governing equations for both the kinematics and the dynamics of the satellite following the Regulation Case [5].

We will use the dynamical formulas on its quaternion version.

3.1. Inertia tensor

First step is to define the tensor of inertia of our spacecraft on its principal axis so that all the products of inertia (non-diagonal parameters) are zero and the inertia tensor becomes diagonal.

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \quad (3.1)$$

After modeling our spacecraft in *SolidWorks* we obtained that in our case an approximation for the inertia tensor values (in $\text{kg}\cdot\text{m}^2$) is the following.

$$I = \begin{bmatrix} 30.31 & 0 & 0 \\ 0 & 85.98 & 0 \\ 0 & 0 & 86.37 \end{bmatrix}$$

Although it is only an approximation and the real inertia tensor would not match the given one, it is a reasonable one for the goal of this project.

3.2. Kinematics

Kinematics focus on the attitude change without considering the forces originating this change.

Let q' be the expression of a triad at time $(t + \Delta t)$ relative to its position at time t with a rotation of $\Delta\phi$ in time Δt .

$$\vec{q}' = \left[e_1 \sin\left(\frac{\Delta\phi}{2}\right), e_2 \sin\left(\frac{\Delta\phi}{2}\right), e_3 \sin\left(\frac{\Delta\phi}{2}\right) \right] \quad q'_4 = \cos\left(\frac{\Delta\phi}{2}\right)$$

Then the relation between the future quaternion, assuming that the satellite is rotating with angular velocity ω , with the actual quaternion results in

$$q(t + \Delta t) = \left[\cos\left(\frac{\Delta\phi}{2}\right) \mathbf{I} + \sin\left(\frac{\Delta\phi}{2}\right) \Omega \Delta t \right] q(t) = \left(\mathbf{I} + \frac{1}{2} \Omega \Delta t \right) q(t) \quad (3.2)$$

Finally, by taking the expression for the derivative of a quaternion

$$\frac{dq}{dt} = \lim_{\Delta t \rightarrow 0} \frac{q(t + \Delta t) - q(t)}{\Delta t} = \frac{dq}{dt} = \lim_{\Delta t \rightarrow 0} \frac{(\mathbf{I} + \frac{1}{2}\Omega\Delta t)q(t) - q(t)}{\Delta t} \quad (3.3)$$

The expression for the quaternion kinematics results in

$$\dot{q} = \frac{1}{2}\Omega(\omega)q \quad (3.4)$$

$$\delta q = q \otimes q_c^{-1} = \begin{bmatrix} \delta \vec{q}_{13} \\ \delta q_4 \end{bmatrix} \quad (3.5)$$

Which is equivalent to compute vector and scalar parts separately as

$$\delta \vec{q}_{13} = \Xi^T(q_c)q \quad (3.6)$$

$$\delta q_4 = q^T q_c \quad (3.7)$$

with

$$\Xi(e) = \begin{bmatrix} e_4 & -e_3 & e_2 \\ e_3 & e_4 & -e_1 \\ -e_2 & e_1 & e_4 \\ -e_1 & -e_2 & -e_3 \end{bmatrix} \quad (3.8)$$

The final expression for the spacecraft's kinematics is

$$\delta \dot{q} = \frac{1}{2}\Omega(\omega)\delta q \quad (3.9)$$

which can be divided into the scalar and vector part of the resulting quaternion.

$$\delta q_{13} = \frac{1}{2}[\delta q_{13} \times] \omega + \frac{1}{2}\delta q_4 \omega \quad (3.10)$$

$$\delta \dot{q}_4 = -\frac{1}{2}\delta q_{13}^T \omega \quad (3.11)$$

3.3. Dynamics

As commented before, kinematics focused on the rotation of the spacecraft without taking into account the torques that generated that rotation. On the other hand, dynamics do consider the torques and momentum changes originating the angular rotation.

Considering the expression for the angular momentum (H) in the body coordinates in terms of the inertia tensor and the angular rates of the spacecraft (ω)

$$H_B = I_B \omega_B \quad (3.12)$$

where L are the external torques.

And its time derivative

$$\dot{H}_B = -\omega_B \times H_B - L_B \quad (3.13)$$

We can obtain the widely known Euler's rotational equation of motion.

$$I\dot{\omega} = -[\omega \times]I\omega + L \quad (3.14)$$

As for this project we will be working with momentum wheels as actuators, we must include their effect into 3.14. That means to add the angular momentum of the wheels ($h \equiv H_B^\omega$) and its time derivative (\dot{h}).

$$I\dot{\omega} = -[\omega \times](I\omega + h) - \dot{h} \quad (3.15)$$

with

$$\dot{h} = -[\omega \times]h - u \quad (3.16)$$

Being u the effective wheel torque input.

We can check that both expressions Eq.3.14 and Eq.3.15 are the same as if we substitute Eq.3.16 in Eq.3.15 results in Eq.3.14.

Finally we must define an expression for the control torque u , which will be discussed in section 5.3.

CHAPTER 4. ATTITUDE DETERMINATION

The fundamental problem of attitude determination is to determine the orientation of the spacecraft's body-frame \mathcal{B} relative to a given reference Cartesian coordinate system \mathcal{R} .

The physical vectors (\mathbf{x}) obtained through the sensors take the form

$$\mathbf{b} = \mathbf{b}^0 + \delta\mathbf{b} = A\mathbf{r} + \delta\mathbf{b} \quad (4.1)$$

where \mathbf{b}^0, \mathbf{r} are the projections of \mathbf{x} on \mathcal{B} and \mathcal{R} respectively, $\delta\mathbf{b}$ is the measurement error committed by the sensor and A is the attitude matrix, which can be determined with multiple sensor observations.

4.1. Wahba's problem

Grace Wahba, in 1965, formulated the attitude problem based on a loss function taking into account all n measurements. The problem is to find A that minimizes

$$L(A) = \frac{1}{2} \sum_{i=1}^n a_i \|\mathbf{b}_i - A\mathbf{r}_i\|^2 \quad (4.2)$$

that is equivalent to maximizing the gain function

$$g(A) = 1 - L(A) = \sum_{i=1}^n a_i \mathbf{b}_i^T A \mathbf{r}_i = \text{tr}(AB^T) \quad (4.3)$$

with

$$B = \sum_{i=1}^n a_i \mathbf{b}_i \mathbf{r}_i^T \quad (4.4)$$

with the constraint that the sum the measurements weights a_i equals one.

$$\sum_{i=1}^n a_i = 1$$

In order to solve Wahba's Problem we will introduce the QUEST algorithm which finds the optimal quaternion associated to the A-matrix that minimizes Eq.4.2 (see, [6] [7]).

4.2. QUEST algorithm

The Quaternion Estimation (QUEST) is an algorithm used to find the optimal quaternion and is associated attitude matrix (A) to solve Wahba's problem. Davenport [8] devised the named *q-method* to find this optimal quaternion by noting that the gain function can be expressed in terms of quaternions as

$$g(A) = \bar{q}^T K \bar{q} = g(\bar{q}) \quad (4.5)$$

with K being the matrix computed as follows

$$S = B + B^T \quad (4.6)$$

$$z = \sum_{i=1}^n a_i \mathbf{b}_i \times \mathbf{r}_i \quad (4.7)$$

$$\sigma = \text{tr}(B) \quad (4.8)$$

$$K = \begin{bmatrix} S - \sigma \mathbf{I}_3 & z \\ z^T & \sigma \end{bmatrix} \quad (4.9)$$

$g(\bar{q})$ is maximized if \bar{q}_{opt} is the eigenvector of K corresponding to its largest eigenvalue, which corresponds to

$$K\bar{q} = \lambda_{max}\bar{q} \quad (4.10)$$

The Cayley-Hamilton theorem states that the eigenvalues ξ of any square matrix satisfies the equation

$$\det(S - \xi I) = 0 \quad (4.11)$$

and specially for a 3×3 matrix, the last equation takes the form

$$-\xi^3 + 2\sigma\xi^2 - k\xi + \Delta = 0 \quad (4.12)$$

with

$$\sigma = \frac{1}{2}\text{tr}(S) \quad k = \text{tr}(\text{adj}(S)) \quad \Delta = \det(S)$$

After this, Eq.4.10 yields in the characteristic equation

$$\lambda^4 - (a+b)\lambda^2 - c\lambda + (ab + c\sigma - d) = 0 \quad (4.13)$$

where

$$a = \sigma^2 - k \quad b = \sigma^2 + z^T z \quad c = \Delta + z^T S z \quad d = z^T S^2 z$$

Although explicit solution could be found to solve for λ it is recommended to avoid it and use the Newton-Raphson method with $\lambda_0 = 1$ as a starting value.

The optimal quaternion is the quaternion associated to the eigenvector of the maximum eigenvalue and is defined as follows (Eq.4.14)

Defining $\omega = \lambda_{max}$ and

$$\alpha = \omega^2 - \sigma^2 \quad \beta = \omega - \sigma \quad \gamma = (\omega + \sigma)\alpha - \Delta$$

being γ the scalar part of the optimal quaternion and

$$X = (\alpha I + \beta S + S^2)z$$

the vector part, the equation for the optimal quaternion is given by

$$\bar{q}_{opt} = \frac{1}{\sqrt{\gamma^2 + |X|^2}} \begin{bmatrix} X \\ \gamma \end{bmatrix} \quad (4.14)$$

The last step of the algorithm is to find the matrix A associated to \bar{q}_{opt} using Eqs. 2.15 or 2.16.

QUEST is a very efficient algorithm to determine the attitude of a satellite, but has the limitation of using just the last determination of the sensors, then losing all the information gathered previously. The use of this previous information would allow a more robust and efficient determination, and this is the goal of the improved QUEST algorithm, the RE-QUEST.

CHAPTER 5. ATTITUDE CONTROL

Once the attitude is well-known after the phase of determination we must generate torque upon the satellite to make it match the desired attitude. In order to do this, in our case, we will use a system of reaction wheels to generate the necessary torque and considering the relatively high orbit of operation, the use of magnetorques will be limited to unload the wheels when saturated.

In this chapter we are going to present the momentum wheel system combinations, the basics of closed-loop (or feedback) control, and briefly describe the main disturbances of the space environment that would affect the attitude of our spacecraft. We will also focus on the assumptions made in this project and finally we will present and discuss the control law.

5.1. Feedback Control

Closed-loop control is a feedback process of sensing the output of a system and applying a control law to make it match a desired or commanded value. Thus, these are processes in which the issued control is dependant of the actual output. In the case of spacecrafts, automatic feedback systems are used in attitude control by constantly determining the output value from the attitude sensors and applying a control torque to bring the satellite to the desired orientation.

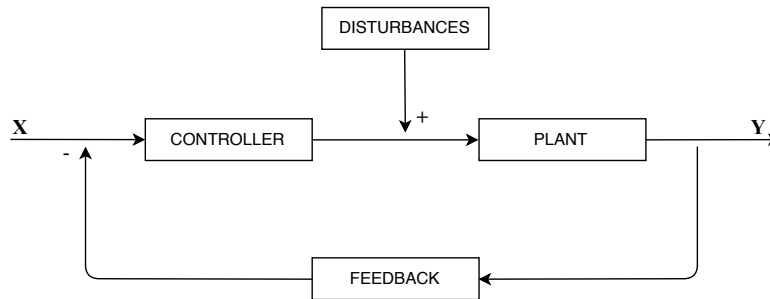


Figure 5.1: Block diagram example.

Those systems are represented by *Block Diagrams* (see Fig.5.1.), schematics associated to the transfer function (Eq.5.1) of the system which is the relation between the systems' output and input expressed as the ratio of two polynomials obtained from the Laplace transform of the satellite's governing equations.

$$\mathcal{H}(s) = \frac{\mathcal{Y}(s)}{\mathcal{X}(s)} = \frac{(s_1 - z_1)(s_2 - z_2) \dots (s_n - z_n)}{(s_1 - p_1)(s_2 - p_2) \dots (s_n - p_n)} \quad (5.1)$$

In the above equation the roots of the numerator are called the zeros of the system and the roots of the denominator are called the poles. The location of the latests determines the stability of the system, as to be stable all the poles must have negative real part.

For the controller, one of the most used mechanisms is the Proportional-Integral Derivative (PID) controller (see section 5.3.).

5.2. Disturbances

Disturbances are torques applied upon the satellite which are external to it, basically caused by the space environment, and which tend to perturb the attitude of the satellite. The disturbances acting on a spacecraft depend directly on its orbital elements, as well as on the body around which it is orbiting.

In our case, the satellite performs a LEO (Low-Earth Orbit) and the possible disturbances that may affect its attitude are explained below.

Disturbance	Dependence on distance from Earth	Region of dominance
Aerodynamic	$e^{-\alpha r}$	$h < 500$ km
Magnetic	$1/r^3$	$500 \text{ km} < h < 35000$ km
Gravity Gradient	$1/r^3$	$500 \text{ km} < h < 35000$ km
Solar Radiation	Independent	Above synchronous altitude

Table 5.1: Environmental Disturbance Torques ([4] p.17)

- **Gravity Gradient Torque**

Any non-symmetric spacecraft is exposed to a gravitational torque due to the variation of force that the Earth's gravitational field apply upon the body.

$$\tau_{gg} = \frac{3GM}{R^3} \vec{a} \times \vec{I} \vec{a} \quad (5.2)$$

This torque can be used as a simple method to stabilize a satellite in a nadir-looking direction, but in our case this will be a perturbation. Given that the torque decreases as R^{-3} , at the altitude of interest its effect will be limited.

- **Solar Radiation Torque**

This environmental torque is produced by the incident Sun radiation on the spacecraft which produces a torque about the satellite's center of mass because of the difference between the incident and reflected flux. The momentum generated depends on the intensity and spectral distribution of the radiation, the optical properties and geometry of the spacecraft's surface and the orientation of the sun. Given that the solar radiation decreases quadratically with the distance from the Sun, solar pressure can be treated as independent from the altitude of the orbit in spacecraft orbiting around the Earth.

$$\tau_{sun} = -P \int r [(1 - C_s) \hat{S} + 2(C_s \cos(\theta) + \frac{1}{3} C_D) \hat{n}] dS \quad (5.3)$$

Even if small, this torque will be prevalent as a perturbation to the ADCS system, as it acts at all time when the satellite is in direct sunlight.

- **Aerodynamic Torque**

Is the torque produced by the interaction of the spacecraft and the upper Earth's atmosphere. As shown in Table 5.1 it is the dominant environmental disturbance for low altitude (< 500 km) orbits.

$$\tau_{aero} = \frac{1}{2} \rho v^2 C_D S \hat{v} \quad (5.4)$$

where ρ is the atmospheric density, v the velocity of the satellite relative to the atmosphere, S its cross-section, l the distance between the center of mass of the satellite and its center of pressure, C_D the drag coefficient, and \hat{v} is a unit vector in the direction of the velocity v .

- **Magnetic Disturbance Torque**

As its name indicates, these are torques generated by the interaction of the Earth's magnetic field with the residual magnetic dipole of the spacecraft. Given the relatively high altitude of the orbit, which accounts for a weak terrestrial magnetic field, and the possibility to reduce the residual magnetic dipole of the satellite by means of judicious design, we do not expect this disturbance to cause any difficulty to the ADCS.

Fig.5.2. shows the torque generated by the aerodynamics torque and the gravitational torque as function of the height of the orbit. As at this stage of the project we do not have the information of the spacecraft's dipole nor the solar cells we cannot compute the torques generated from magnetic and solar radiation respectively.

In Fig.5.2. the green line represents the gravitational torque which is constant for all computed altitudes. The rest of lines in the figure represent the following configurations of the aerodynamic torque; in purple the aerodynamic torque for maximum solar activity (index for solar flux $10.7=250$) at noon, in yellow maximum solar activity at midnight, in red for minimum solar activity (index for solar flux $10.7=50$) at noon and finally in blue for minimum solar activity at midnight. As we can observe in Fig.5.2. the torques are of low magnitude at high orbits and thus can be neglected for our numerical simulations.

5.3. PID controller

A PID controller is the most used control algorithm as it is highly robust, can be used in a wide range of conditions and it is simple to understand and use.

With the use of a PID controller, the optimal response is found by tuning each of the three components and finding the combination that best fits the goal in each case.

- **Proportional Gain**

The increase of the proportional gain will speed up the control system response but

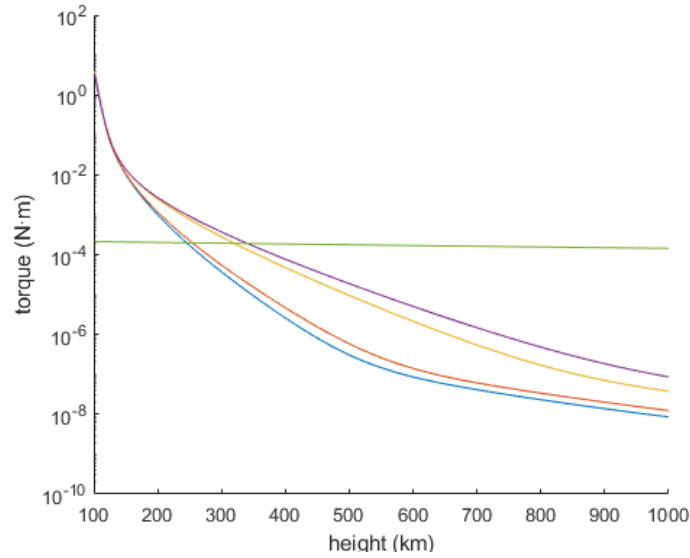


Figure 5.2: Gravitational and aerodynamic torques as function of orbit altitude

an excessive gain will generate oscillations and even make it become unstable.

Proportional gain changes the output in proportion to the error so if the error gets bigger so does the control action following

$$\text{Control Action} = \text{Proportional Gain} \times \text{Error}$$

- **Integral Gain**

The function of the integral gain is to increment or decrement the action of the controllers in order to drive the steady-state error to zero.

- **Derivative Gain**

The derivative gain adjusts the slope of the error over the time so it modifies the rate of change of the controller, increasing the stability and reducing oscillations and overshoot of the response.

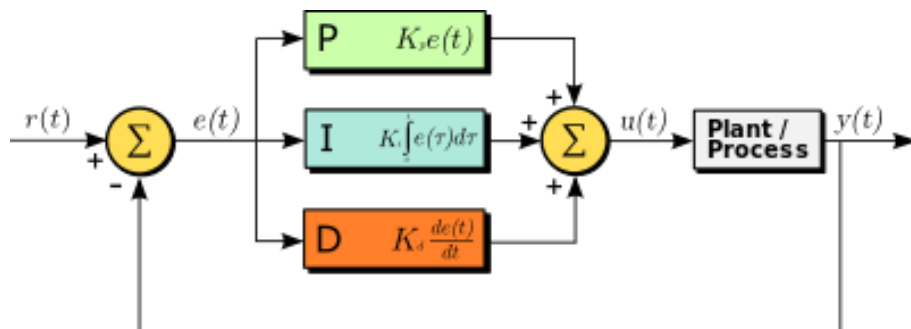


Figure 5.3: PID controller Block Diagram

5.3.1. Control Law

In our case, we will use an only PD controller, that is without integral part as we think it would be enough to reach our control goals. In terms of quaternions, the goal of the controller is to bring the error quaternion δq to the identity $q_I = [0 \ 0 \ 0 \ 1]^T$ and nullify the angular rate ω .

$$u = -k_p \text{sign}(\delta q_4) \delta q_{13} - k_d \omega \quad (5.5)$$

Once defining the values for the proportional and derivative gains (positive scalars) we must take into account that the torque generated in Eq.5.5 must be below the limits of the momentum wheels.

The proposed control law (Eq.5.5) introduced into Eq.3.15 results in the closed-loop system governed by kinematics equation Eq.3.9 and

$$\dot{\omega} = -I^{-1}([\omega \times]I\omega + k_p \delta q_{13} + k_d \omega) \quad (5.6)$$

As the inertia tensor I is non-zero, the only equilibrium point, that is the values that bring the rates to zero, is found by having interior of the parenthesis equal to zero. This only happens if the vector part of the error quaternion and the angular rates are zero.

$$\delta q_{13} = 0$$

$$\omega = 0$$

5.3.2. Lyapunov Stability test

Dynamical systems are mathematically described by means of a differential equation that captures the relevant Physics involved in the problem. This differential equation, in general a non-linear one, can be used to analyse the stability of the system in front of disturbances and control inputs.

A useful method is the Lyapunov direct stability analysis. A system is said to be Lyapunov stable when the dynamical equation describing it

$$\dot{s} = f(s, u) \quad (5.7)$$

where s is the state vector defining the system and u are the control inputs, verifies that if $s_0 \in [s' - \delta, s' + \delta]$ at $t = 0$, it remains so for every $t > 0$. It can be proved that this is equivalent to state that there exists a function (called Lyapunov function) $V(s)$ such that

$$V(s) : \mathbb{R}^n \Rightarrow \mathbb{R} \quad (5.8)$$

and verifies that

$$V(s) = 0 \Leftrightarrow s = 0 \quad (5.9)$$

$$\dot{V}(s) = \sum_{i=0}^{\infty} \frac{\partial V}{\partial s_i} \dot{s}_i \leq 0 \quad \forall s \neq 0 \quad (5.10)$$

In Classical Mechanics, this Lyapunov function can be readily identified with the potential of the system, which is stable in precisely the conditions stated just above. Nevertheless, Lyapunov function are more general than classical potentials, and provide a much wider usability for determining the stability of a dynamic system under disturbances and control actuation.

Following Landis [5], we propose the following Lyapunov function based on the error quaternion $\delta \mathbf{q}$:

$$V = \frac{1}{4} \omega^T I \omega + \frac{1}{2} k_p \delta \mathbf{q}_{1:3}^T \delta \mathbf{q}_{1:3} + \frac{1}{2} k_p (1 - \delta q_4)^2 \quad (5.11)$$

whose time derivative is

$$\dot{V} = \frac{1}{2} \omega^T I \dot{\omega} + k_p \delta \mathbf{q}_{1:3}^T \delta \dot{\mathbf{q}}_{1:3} - k_p (1 - \delta q_4) \delta \dot{q}_4 \quad (5.12)$$

which reduces to

$$\dot{V} = -\frac{1}{2} \omega^T V \omega \leq 0 \quad (5.13)$$

which proves that the regulation problem is Lyapunov-stable.

CHAPTER 6. RESULTS AND ANALYSIS

Numerical simulations have been made using the data of the example of [6] for the observation vectors and [5] example 7.1 for the angular rates.

Hereafter in this section we are going to analyze the response of the system in terms of the quaternion, the error quaternion, the angular rates and the applied torque and how the proportional and derivative gains affects.

We desire to have a response as fast as possible but avoiding oscillations. After checking several combinations of values for the proportional and derivative gains, the one that best fits our requirements is having a $k_p = 1$ and $k_d = 15$ (Fig.6.1). The "high" value of the derivative part with respect to the proportional one increases stability and avoid the possible oscillations.

As we can see in Fig.6.1 the quaternion meets the desired values at around 100 seconds and at this time the angular rate is already brought to zero. Regarding the torques applied, we can see how for the given case small torques (below $1Nm$) are necessary.

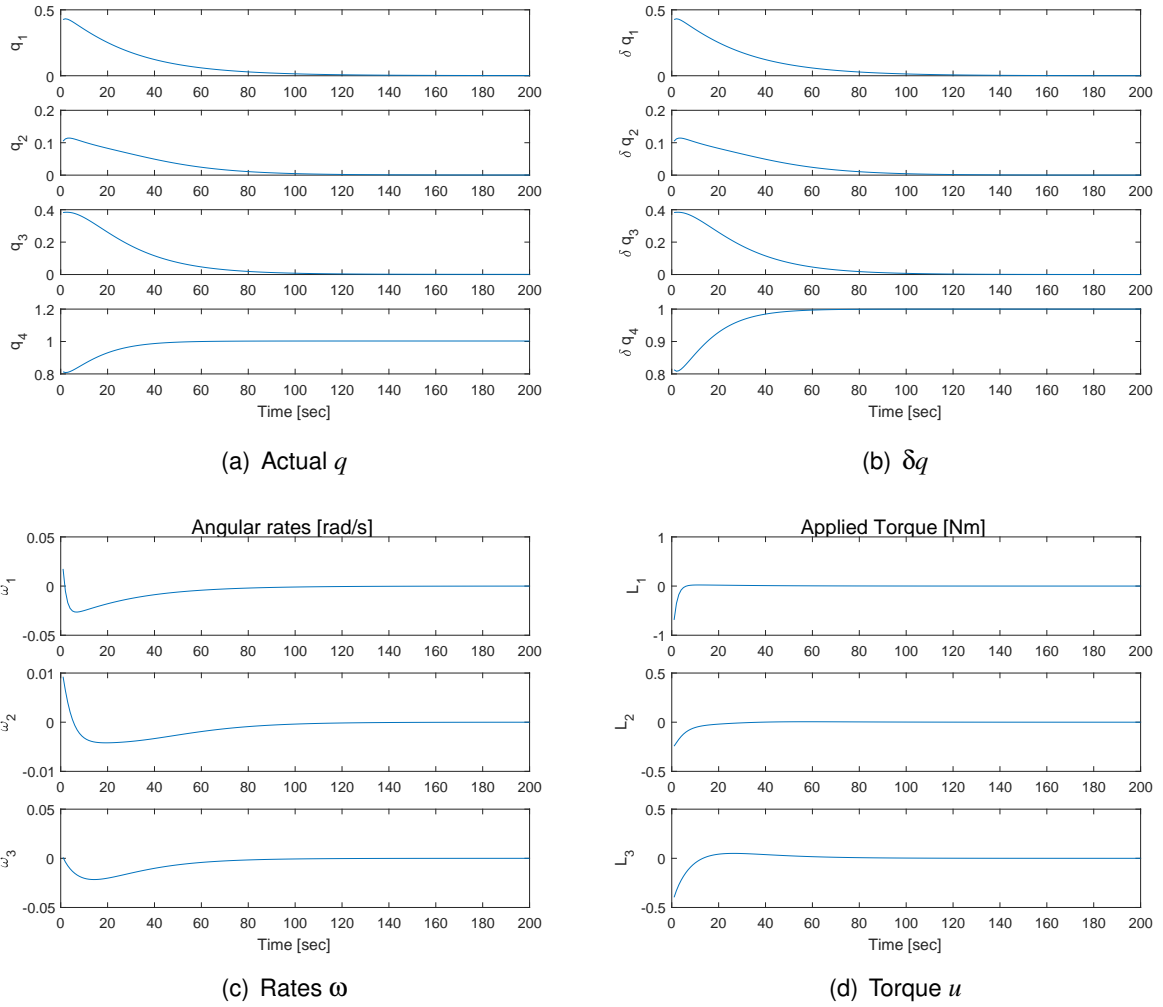
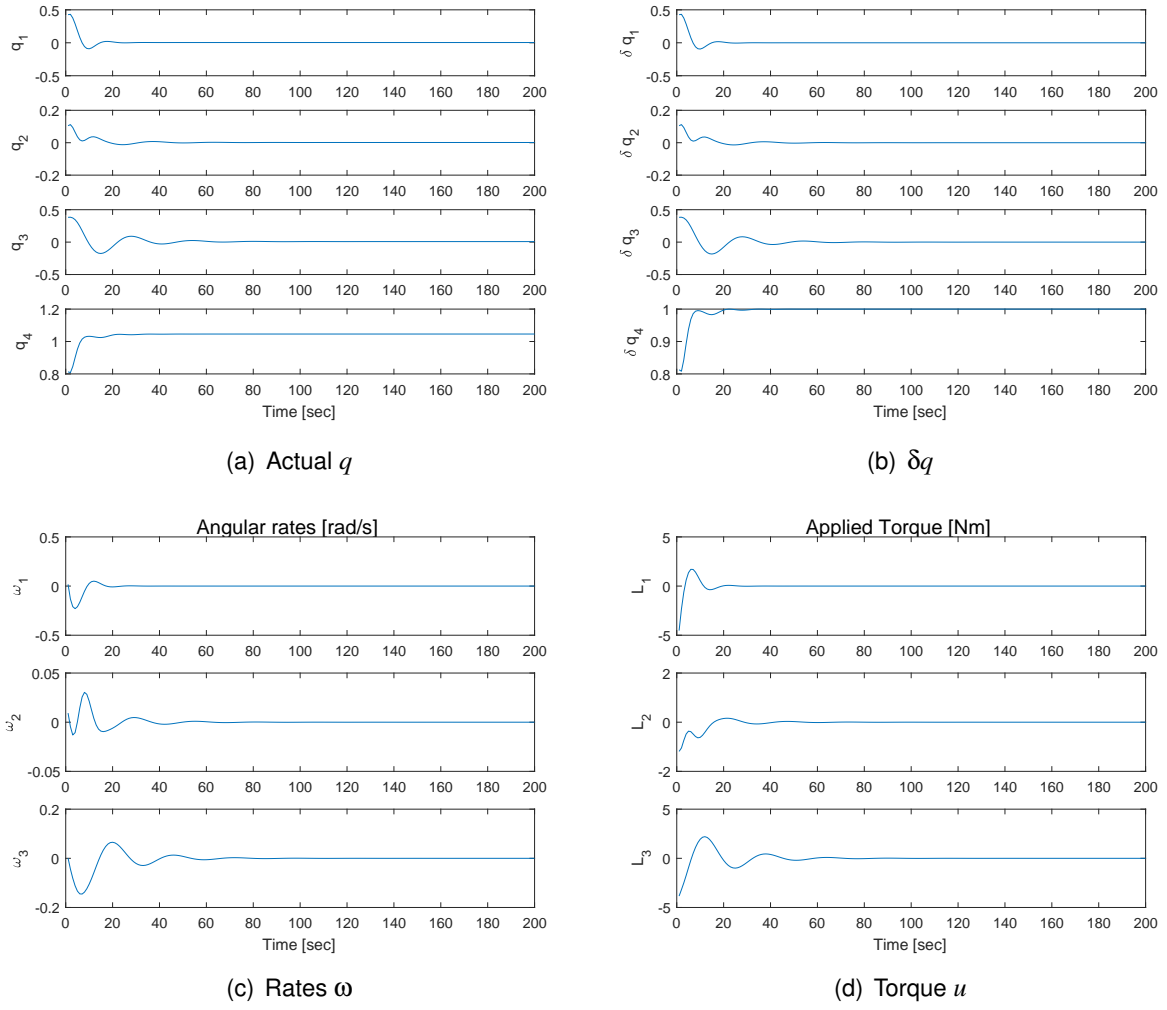


Figure 6.1: Simulations results for $k_p = 1$ and $k_d = 15$

Fig.6.2 shows how by increasing the proportional gain and maintaining the same derivative part the response becomes faster but with the appearance of overshoot which is not desirable in our case.

Figure 6.2: Simulations results for $k_p = 10$ and $k_d = 15$

After the appearance of the overshoot, if we give more importance to the proportional gain than to the derivative gain the response becomes oscillatory and unstable and thus, never sets to the final value (Fig. 6.3).

From here we can determine that the derivative gain must be higher than the proportional gain.

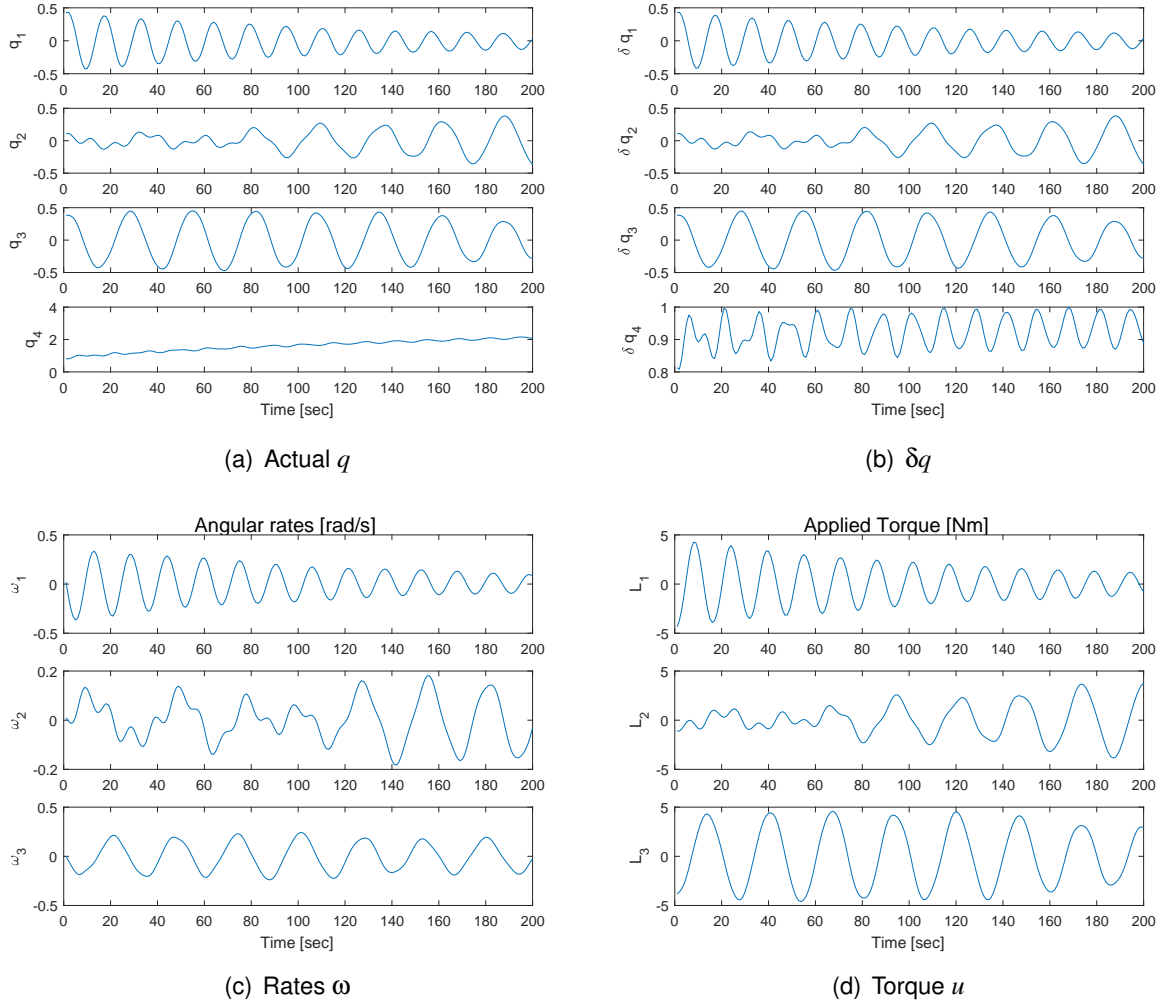
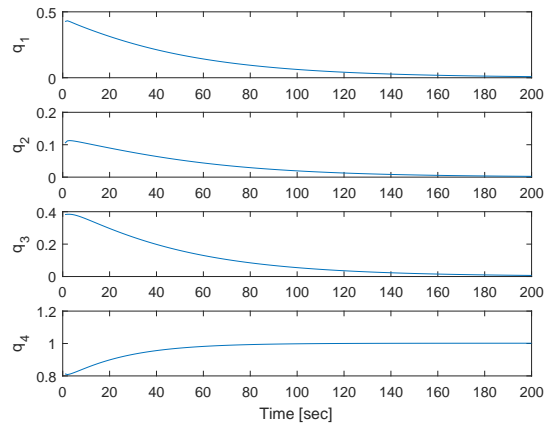
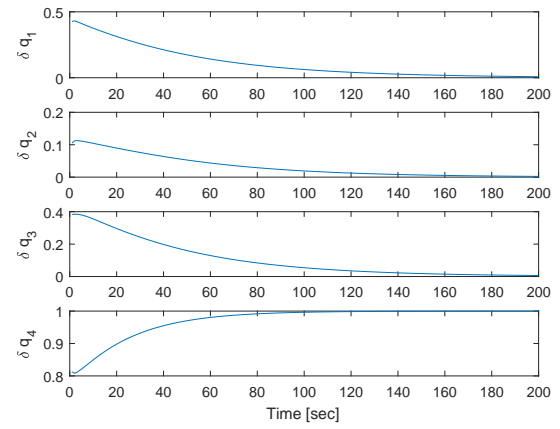
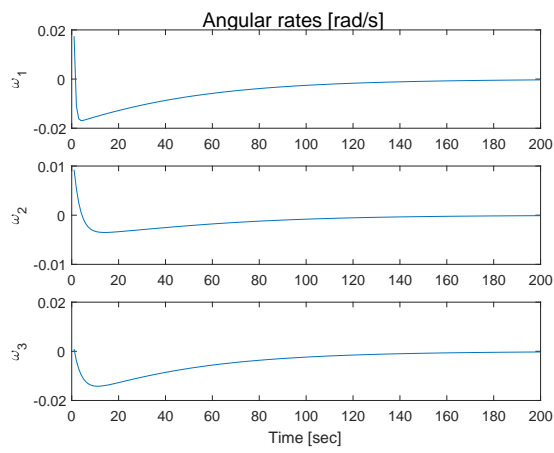
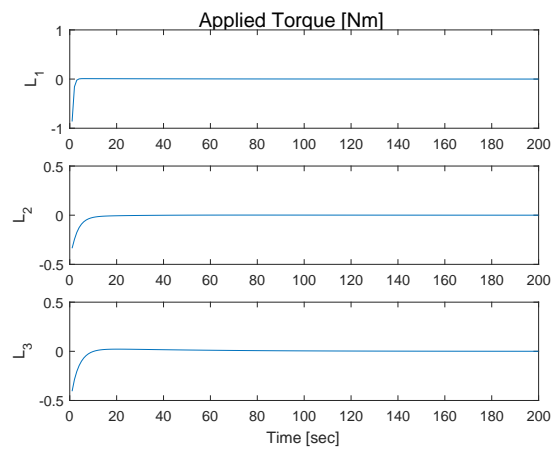


Figure 6.3: Simulations results for $k_p = 10$ and $k_d = 5$

However we must consider that a higher derivative part with a low proportional will make our system to be very stable and avoid oscillations but will also make it slower. As shown in Fig. 6.4 the response presents no oscillations nor overshoot but sets at 200 seconds which is twice the time that for the considered as good option (Fig. 6.1).

(a) Actual q (b) δq (c) Rates ω (d) Torque u Figure 6.4: Simulations results for $k_p = 1$ and $k_d = 25$

CONCLUSIONS

The aim of this project was to check the viability of designing a quaternion based ADCS for a small space telescope. Through an extensive research on scientific literature we have obtained the suitable equations for describing the kinematics and dynamics in quaternions of a three axis stabilized spacecraft using momentum wheels as principal actuators.

First of all we have carried out a preliminary design of the hST using SolidWorks and we have obtained a first version of the inertia tensor used to perform the numerical simulations.

With the obtained equations we have designed a QUEST algorithm to solve Wahba's attitude problem and applied a PD controller to control the necessary torque at each moment to bring the spacecraft's attitude to the desired one for observation. The correct functioning of the programmed algorithm have been checked using numerical examples from the different references used during this project.

Finally, we have substituted the tensor of inertia of the examples for our spacecraft's tensor and checked that with the correct gains the goal of this project is achieved and thus the spacecraft can be correctly controlled with the purposed systems.

Future work on the determination segment could include the use of REQUEST or optimal REQUEST. In both cases, the algorithms are based on the QUEST (already implemented) but historical data are used along with the new attitude determination observations gathered by the sensors. Then, attitude control can be more accurate. Optimal REQUEST differs from REQUEST on its use of an optimized value for the damping function (that reduces the relevance of older attitude determinations in order to avoid undesirable effects).

FUTURE WORK

The following steps to work on this project are extensive as it has been a first approach on the theoretical part and the check of viability of the suggested system for the ADCS.

First of all, for the attitude determination the QUEST algorithm should be improved to introduce the REQUEST algorithm which updates matrix K with the new measurements done by the sensors or even an Optimal-REQUEST which convenience should be analysed.

Secondly, the gains of the PID in this work have been selected empirically but as future work those can be studied to obtain an optimal response or computed through the Monte-carlo method.

Finally, after having computed an approximation of the conditions in which our spacecraft will behave, the different components (sensors and actuators) must be chosen according to the torque and accuracy demands and the weight and cost limitations.

Furthermore, as the project done here focuses on the ADCS subsystem of the global hST project, the others subsystems must be studied to finally reach the goal of presenting a complete project that might sometime in the future see the light and succeed.

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APPENDICES

APPENDIX A. MATLAB CODES

A.1. Main program

```
1 %clear all;
2 close all;
3 %% Data measurements on Body and Reference frames
4 %Example values from REQUEST paper
5 r1=[0.267 0.535 0.802]';
6 r2=[-0.667 -0.667 -0.333]';
7 r3=[0.267 -0.802 0.535]';
8 r4=[-0.447 0.894 0.000]';
9
10 b1=[0.688 0.662 0.297]';
11 b2=[-0.985 -0.120 -0.123]';
12 b3=[-0.280 -0.030 0.959]';
13 b4=[0.303 0.575 -0.760]';
14
15 std1=0.01;
16 std2=0.05;
17 std3=0.03;
18 std4=0.02;
19
20 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
21 % DA=[0.936 -0.283 -0.210;0.303 0.951 0.068;0.181 -0.127 0.975];
22 % b2=DA*b2;
23 % b1=DA*b1;
24 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
25 % W=[b1 b2 b3 b4];
26 % V=[r1 r2 r3 r4];
27 % stddev=[std1 std2 std3 std4];
28
29 %%First two pairs of vectors
30 W=[b1 b2];
31 V=[r1 r2];
32 %Weights vector
33 stddev=[std1 std2];
34
35
36 %% Space Telescope data
37 % Intertia matrix and its inverse
38 %J=[];
39 %invJ=inv(J);
40 % Initial angular rate in rad/s
41 %w=[];
42
```

```

43
44 %% QUEST Algorithm
45 [A2,q,lambda2]=QUEST(W,V,stddev);
46 % Where q is the actual quaternion computed with QUEST algorithm
47
48
49 %% CONTROL LAW
50
51 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
52 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
53 %Values from Landis example 7.1: Regulation Case
54 q=[0.685;0.695;0.153;0.153];q=q/norm(q);
55 w=[0.53;0.53;0.053]*pi/180;
56 J=diag([10000 9000 12000]);invJ=inv(J);
57 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
58 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
59 % Desired Quaternion
60 qd=[0 0 0 1];
61 % Define proportional (kp) and derivative (kd) gains
62 kp=50;
63 kd=500;
64 [Qmat,Wmat,Umat,Qerr]=control(J,invJ,w,q,qd,kp,kd);
65
66
67 %% PLOTS
68 t=[1:500];
69 %Actual quaternion
70 figure;
71 set(gca,'fontsize',10)
72 subplot(4,1,1);plot(t,Qmat(:,1));ylabel('q_1');
73 subplot(4,1,2);plot(t,Qmat(:,2));ylabel('q_2');
74 subplot(4,1,3);plot(t,Qmat(:,3));ylabel('q_3');
75 subplot(4,1,4);plot(t,Qmat(:,4));ylabel('q_4');xlabel('Time [sec]
    ');
76 %Error Quaternion
77 figure;
78 set(gca,'fontsize',10)
79 subplot(4,1,1);plot(t,Qerr(:,1));ylabel('\delta q_1');
80 subplot(4,1,2);plot(t,Qerr(:,2));ylabel('\delta q_2');
81 subplot(4,1,3);plot(t,Qerr(:,3));ylabel('\delta q_3');
82 subplot(4,1,4);plot(t,Qerr(:,4));ylabel('\delta q_4');xlabel('
    Time [sec]');
83 %Angular Rate
84 figure;
85 subtitle('Angular rates [rad/s]');
86 set(gca,'fontsize',10)
87 subplot(3,1,1);plot(t,Wmat(:,1));ylabel('\omega_1');
88 subplot(3,1,2);plot(t,Wmat(:,2));ylabel('\omega_2');

```

```
89 subplot(3,1,3);plot(t,Wmat(:,3));ylabel('\omega_3');xlabel('Time  
[sec]');  
90 %Torque  
91 figure;  
92 subtitle('Applied Torque [Nm]');  
93 set(gca,'fontsize',10)  
94 subplot(3,1,1);plot(t,Umat(:,1));ylabel('L_1');  
95 subplot(3,1,2);plot(t,Umat(:,2));ylabel('L_2');  
96 subplot(3,1,3);plot(t,Umat(:,3));ylabel('L_3');xlabel('Time [sec]  
' );
```

A.2. QUEST Algorithm function

```
1 %% QUEST ALGORITHM
2
3 %% Introduction
4 %Wahba's Problem:
5 %Find optimal attitude matrix A that minimizes
6 %sum||b(i)-Ar(i)||^2 with the constraints that A'*A=I3 and det(A)
   =1
7
8 %W= Projection of vector of measurements to B-frame
9 %V= Projection of vector of measurements to R-frame
10 % Length of W,V are the number of simulations
11
12 %%
13 function [A,qopt,lambda]=QUEST(W,V,stddev)
14
15 %Constraint that the sum of the weights must be 1.
16 mk=sum(stddev.^(-2));
17 weights=stddev.^(-2);
18
19 B=(weights/mk).*W*V';
20 S=B+B';
21 z=sum((weights/mk).*cross(W,V),2);
22 sigma=trace(B);
23 K=[S-sigma*eye(3) z;z' sigma];
24 adjS=det(S)*inv(S);
25 k=trace(adjS);
26
27 %Characteristic equation: p(x)⟶ x^4-(a+b)x^2-c*x+(a*b+c*sigma-d
   )=0
28 a=sigma^2-k;
29 b=sigma^2+z'*z;
30 c=det(S)+z'*S*z;
31 d=z'*S*S*z;
32
33 %%Explicit solution for the eigenvalues
34 % polyeq=[1,0,-a-b,-c,a*b+c*sigma-d];
35 % egn=roots(polyeq);
36 %%Eigenvalues can also be calculated from matrix K
37 % egnK=eig(K);
38 % %Maximum Lambda
39 % maxlam_explicit=max(abs(egnK));
40
41 %Newton-Raphson Theorem to find lamda max.
42 %QUEST paper sais explicit solution is desired to be avoided.
43 lambda=1;
44 lamprev=0;
```

```

45 count=0;      %Counter for the number of iterations performed in
               the loop.
46 while abs((lambda-lamprev)/lambda)>=1E-12
47     lamprev=lambda;
48     lambda=lambda -(lambda^4-(a+b)*lambda^2-c*lambda+(a*b+c*sigma
               -d))/(4*lambda^3-2*(a+b)*lambda-c);
49     count=count+1;
50 end
51 maxlam=lambda;
52
53 %Optimal Quaternion
54 omega=maxlam;      %Assume omega is equal to maximum lambda
55 alpha=omega^2-sigma^2+k;
56 beta=omega-sigma;
57 gamma=(omega+sigma)*alpha-det(S);
58 X=(alpha*eye(3)+beta*S+S*S')*z;
59
60 qopt=[X;gamma];      %Quaternion=[Vector part ; Scalar part]
61 qopt=qopt/norm(qopt);
62 q1=qopt(1);q2=qopt(2);q3=qopt(3);q4=qopt(4);
63 qvect=[q1 q2 q3]';
64 %Attitude Matrix
65
66 Q=[ 0 -q3 q2;
67     q3 0 -q1;
68     -q2 q1 0];
69
70 A=(q4^2-(norm(qvect)^2))*eye(3)+2*(qvect*qvect')-2*q4*Q;
71
72 %Which is equivalent to:
73 A2=[q1^2-q2^2-q3^2+q4^2      2*(q1*q2+q3*q4)      2*(q1*q3-q2*q4
       );
74     2*(q1*q2-q3*q4)      -q1^2+q2^2-q3^2+q4^2      2*(q2*q3+q1*q4)
       ;
75     2*(q1*q3+q2*q4)      2*(q2*q3-q1*q4)      -q1^2-q2^2+q3
       ^2+q4^2];
76 end

```

A.3. Control function

```
1 function [Qmat,Wmat,Umat,Qerr]=control(J,invJ,w,q,qd,kp,kd)
2 % Qmat is a matrix containing the evolution of the actual
   quaternion
3 % Wmat is a matrix containing the evolution of the angular rate
4 % u is a matrix with the applied torque
5 %%%%%%%%%%%%%%BEFORE CHANGES
6 %Error quaternion
7 invqd=[-qd(1:3) qd(4)];
8
9 q=q';
10 Qmat(1,:)=q;
11 Wmat(1,:)=w;
12
13 %Max time
14 T=500;
15 for i=1:T
16     Xi=[qd(4) -qd(3) qd(2);
17         qd(3) qd(4) -qd(1);
18         -qd(2) qd(1) qd(4);
19         -qd(1) -qd(2) -qd(3)];
20     qerr13=Xi'*q';
21     qerr4=q*qd';
22     qerr=[qerr13;qerr4];
23 %Control Law
24 u=-kp*sign(qerr(4))*qerr(1:3)-kd*w;
25
26 %Angular rate matrices
27 wx=[0 -w(3) w(2)
28     w(3) 0 -w(1)
29     -w(2) w(1) 0];
30
31 Omega= [0 w(3) -w(2) w(1);
32         -w(3) 0 w(1) w(2);
33         w(2) -w(1) 0 w(3);
34         -w(1) -w(2) -w(3) 0];
35 %Kinematics
36 knm=0.5*Omega*q';
37 %Dynamics
38 dyn=-invJ'*wx*J*w+invJ*u;
39
40 %Create matrices
41 Qerr(i,:)=qerr;
42 Umat(i,:)=u;
43
44 if (i<T)
45     Qmat(i+1,:)=Qmat(i,:)+knm';
```

```
46 Wmat( i + 1 ,:)=Wmat( i ,:)+dyn ' ;
47 end
48
49 %Update parameters
50 q=q+knm ' ;
51 q=quatnormalize (q) ;
52 w=w+dyn ;
53 end
54 end
```